

Neurogeometry of color vision

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Abstract

In neurogeometry, principles of differential geometry and neuron dynamics are used to model the representation of forms in the primary visual cortex, V1. This approach is well-suited for explaining the perception of Illusory contours such as Kanizsa's figure (see [Petitot \(2008\)](#) for a review). In its current version, neurogeometry uses achromatic inputs to the visual system as the starting-point for form estimation. Here we ask how neurogeometry operates when the input is chromatic as in color vision. We propose that even when considering only the perception of form, the random nature of the cone mosaic must be taken into account. The main challenge for neurogeometry is to explain how achromatic information could be estimated from the sparse chromatic sampling provided by the cone mosaic.

This article also discusses the non-linearity involved in a neural geometry for chromatic processing. We present empirical results on chromatic discrimination to illustrate the geometric complexity of the discrimination contour when the adaptation state of the observer is not conditioned. The underlying non-linear geometry must conciliate both mosaic sampling and regulation of visual information in the visual system.

Keywords: Neurogeometry, Color vision, Cone mosaic, Chromatic discrimination, Quantum vision

1. Introduction

The relation between the physiology of the visual system and the phenomenology of visual appearance has been recently readdressed by the principles of neurogeometry ([Petitot, 2008](#)). Neurogeometry defines V1 as a contact bundle of 1-jets ([Hoffman, 1985](#); [Koenderink and Van Doorn, 1987](#); [Petitot, 2003](#)) in which neuron dynamics are shown to be geodesics. The main idea behind is that neural activity in response to physical stimuli provides a particular geometry in which perception arises. For instance, this framework takes advantage of the orientation selectivity in primary visual cortex V1 to explain illusory contours such as Kanizsa's figure. [Petitot \(2003\)](#) further assumes that visual appearance is an isomorphism of this emerging neural level bridging the gap between low-level neural activity and high-level visual perception.

A great advance for vision scientist would be to understand how the response of photoreceptor cells lead to visual perception as this would constitute a large step toward the design of efficient assistive devices. As we will see in the following sections, neurogeometry could be a key to improving current devices and allowing implanted people to recover more than a few phosphenes ([Zrenner, 2002](#)).

As it stands, neurogeometry does not take into account the chromatic input at the photoreceptor level. This is contrary to our current understanding of daylight vision. It is usually thought that in photopic condition only cones are functioning in the retina. Rods are fully saturated and are unable to encode incoming stimuli ([Baylor et al., 1984](#)) (see [Bouman \(2006\)](#) for an alternative view). In ad-

dition of being the source of color vision, the cones' are also the primary source of spatial information. The cones' encoding of achromatic information (i.e. quantum catch average over the visible spectrum) depend on their different sensitivities to wavelength¹. So, how is achromatic information for the perception of form, and consequently the neurogeometry of form, estimated using the trichromatic cones' system?

We cannot consider² that the retinal map $\mathbf{R} \subset \mathbb{R}^2$ is extended by the chromatic domain $\mathbf{C} \subset \mathbb{R}^3$ such that the input for the visual system is the domain spanned by $\mathbf{R} \times \mathbf{C}$. Because there is only one possible cone type at each location $\mathbf{R} \times \mathbf{C}$ does not exist at the photoreceptor level. Instead the information is a spatial multiplexing of chromatic information that is more like \mathbf{R} , having the dimension of \mathbb{R}^2 .

Another difficulty is that cones of different types are randomly arranged in the cone mosaic such that a strong dependency exist between color and form at the scale of a few photoreceptors ([Alleysson, 2010](#)). Randomly arranged cone types means that the local chromatic neighborhood changes from place to place in the retina. This is a problem for neurogeometry because it is built upon generalized lin-

¹Cones are of three types, called L-, M- and S-cone for their sensitivities to Long, Medium and Short range of wavelength light respectively

²As [Petitot \(2008\)](#), we called the retinal map \mathbf{R} , which consists of any position $(x, y) \in \mathbb{R}^2$ in the retina. Because of trichromacy nature of color vision we may assume $\dim(\mathbf{C}) = 3$. \mathbf{C} is a set of three numbers representing each cone type. $\mathbf{R} \times \mathbf{C}$ is putting the three cone values \mathbf{C} at each point \mathbf{R}

ear operators (such as sophisticated wavelets [Candès and Donoho \(1999\)](#); [Mallat and Peyré \(2007\)](#)) and neuron dynamics that apply locally to the global network of interconnected cells. The application of neurogeometry to non uniform neighborhoods is still a mostly unexplored topic ([Daubechies et al., 1999](#)). Two scenarios may happen.

1. The visual system could build a map of unambiguous achromatic and chromatic information from the mosaic. Because neurogeometry applies at the level of V1 this map must be constructed in between photoreceptors and the primary visual cortex and need to be compatible with the functional architecture in V1. Thus, would be expected to be consistent with our current models, describe V1 as a hypercolumn ([Hubel and Wiesel, 1977](#)) or as localized, band-pass oriented receptive fields ([Field, 1987](#); [Olshausen et al., 1996](#); [Bell and Sejnowski, 1997](#)). Neurogeometry agrees well with hypercolumn. However, cone mosaic sampling is unrelated to the properties of V1 receptive fields when modelled through redundancy reduction using a statistical analysis of the spatio-chromatic pattern of natural scene ([Alleysson and Süsstrunk, 2004a,b](#); [Alleysson, 2010](#)).
2. Alternatively, the visual system may not need such an unambiguous map. Operators and neural dynamics for vision could be formalized independently of the non uniformity of the local cone neighborhoods. Indeed ([Petitot, 2008](#)), operators are like geodesics in a variational model ([Mumford, 1994](#)) and neural network dynamics are like oscillators ([Hodgkin and Huxley, 1952](#)). The difficulty with this scenario is that the spatial multiplexing of chromatic components induced by the mosaic can be thought of as a high frequency modulation of a scenes spatial information ([Alleysson et al., 2005](#)). It seems that both approaches would encounter difficulties when dealing with this high frequency component. Still, operators or dynamics could include a compensatory element (i.e. a normalizing factor depending on the non uniformity) or operate in a dimension "orthogonal" to the dimension in which the non uniformity is appearing. But, as far as we know, the response of neighboring photoreceptors of different kind should be compared if we wish for color discrimination and vision ([Calkins and Sterling, 1999](#)). Such comparison process is at odds with an orthogonality between retinal map and visual processing.

We previously showed that a nonlinear process of renormalization must operate when considering the trichromatic mosaic sampling by the cones as input to the Parvocellular pathway ([Alleysson, 2010](#)). We review the evidences for such a renormalization process in Section 2. In Section 3, we recall other non-linearities associated with the processing of light intensity. The two scenarios mentioned above are discussed in section 4. We also speculate

on a possible common origin for the two kinds of non-linearities, as reviewed in Sections 2 and 3.

2. Sampling through the mosaic of cones

The human retina contains two kinds of photoreceptor cells, rods and cones. The rod and cone cells span the much of the dorsal surface of the retina and are responsible for the transduction of visual information. Rod cells are highly sensitive to light and it is believed that only a few photons are enough to produce a sensation of flash light ([Baumgardt, 1948](#); [Hecht et al., 1942](#); [Velden, 1944](#)). The drawback of this high sensitivity is that the rod cells are saturated in daylight (photopic) vision and are unable to inform about the achromatic contents of the environment ([Baylor et al., 1984](#)). To the contrary, cones are less sensitive than rods and are functioning in daylight condition. Thus, achromatic spatial information must be constructed from the response of the three cone types.

The topology of the human retinal mosaic has been described by ([Osterberg, 1935](#)). It was found that cone cells are more numerous in the fovea where the spatial acuity is the highest and their density decrease with eccentricity. To the contrary, rods are absent in the fovea and their density increase with eccentricity. It is well known that color vision in human must be mediated by the differential responses of the three cone types ([de Valois and de Valois, 1993](#); [Dacey, 1999](#)). But in addition, the fact that cones are more numerous in the center of the fovea, where the acuity is the highest, suggests that cones are also used for spatial vision.

Another topological curiosity is that the three types of cones are arranged in a random mosaic. This view is supported by studies using adaptive optics of the retina ([Rororda and Williams, 1999](#)) as shown in Figure 1(a) as well as direct microspectrophotometric measurements ([Mollon and Bowmaker, 1992](#)). Genomics ([Hagstrom et al., 1998, 2000](#); [Knoblauch et al., 2006](#)) propose that the disposition of the cone on the mosaic follow a stochastic process regulated by DNA. To illustrate the effect of the mosaic, we simulated the way retinas samples a natural image. For this simulation, shown in Figure 1(b,c), we assimilated the RGB values of a color image to the LMS values of the cone's response to radiation. As discussed previously ([Alleysson, 2010](#)), such a short cut is only valid when considering the effects of different mosaics on information representation.

Because the cone mosaic is random, the local neighborhood changes across the retina and from individual to individual. This implies that a post-receptor cell - when acting as a linear operator for estimating any component of visual information - would need to adapt to the local arrangement of cones in the mosaic. This problem is particularly challenging for neurogeometry since its aim is to describe a global behavior from a multitude of identical local processing units. As already discussed above, the

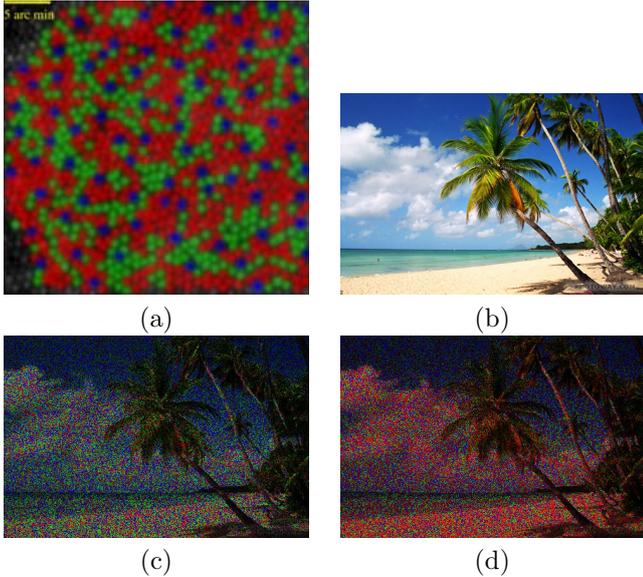


Figure 1: (a) Adaptive optics micrograph of the human retina shows the random arrangement of cone types in the mosaic (From [Roorda and Williams \(1999\)](#)) (b) An RGB image (c) Simulation of the sampling of a color image by a random mosaic of chromatic samples with equal proportion of L-, M- and S-cone types (d) Simulation of another observer (or another part of the retina of the same observer) with a larger proportion of L-cone.

operators or dynamics involved in neurogeometry are not directly applicable to the random chromatic mosaic.

Usually, this problem is thought to be solved by the saccadic movement of the eye ([Maloney and Ahumada, 1989](#)). Even if there is a single chromatic component that is sampled at each spatial position, the micro-saccades, drifts and tremors of the eye should allow several different chromatic components to be sampled at any position of the visual scene. Simulation based on information theory show a close relationship between eye movement and cone density ([Lewis et al., 2003, 2004](#)). However, the relationship between eye movement and the local chromatic neighborhood have never been studied so far. How does a postreceptoral cell deal with the temporal succession of different chromatic events? The problem involved by this temporal-chromatic multiplexing is the same than the one found in spatial-chromatic multiplexing. It is unclear how temporal sampling through saccade could be used to solve the mosaic problem. Actually, the question is rather difficult to address. In one side there are evidences that kinesis is a fundamental trick for explaining vision ([Berthoz and Petit, 2006; Berthoz et al., 2008](#)), ([Petitot \(2008\)](#) pp. 382-390) and in the other side there is no demonstration that color enter in its category of phenomenon.

To examine the hypothesis that the visual system is able to reconstruct spatial and chromatic information from the mosaic before primary visual cortex V1, we developed a model of information representation in a chromatic mosaic ([Alleysson, 2010; Alleysson et al., 2005](#)). This model

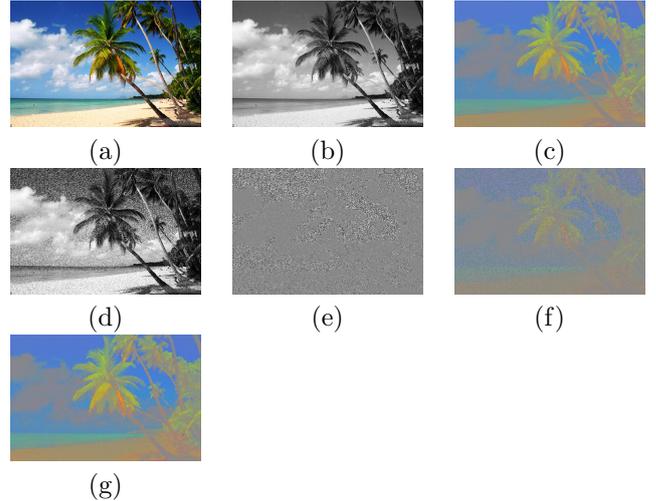


Figure 2: Decomposition of a color image (a) into its achromatic (b) and chromatic (c) parts. When considering a mosaic image simulating cone responses (d) (The image is displayed in B&W because we do not a priori know the type of cone at a particular position but only its response, independently of its type) we can subtract the achromatic information (b), resulting in image (e). The demultiplexing of that image by associating a color to the sampling lattice of each type of cone gives (f), which after interpolation (g) gives an estimate of the chromatic component of (c). This simulation shows that a mosaic image could be decomposed into fully resolved achromatic part plus a subsampled and multiplexed chromatic part.

shows that a mosaic image³ is a sum of the achromatic and chromatic components of the corresponding color image. The achromatic part is spatially fully represented, at the maximal spatial resolution of the sampling lattice since every cone samples at least some achromatic information. However, achromatic information cannot inform about the chromatic content of the visual scene because it is an orthogonal projection inside the cone's color space. In contrary, the chromatic information is subsampled and modulated in a way which correspond to the mosaic pattern (see Figure 2).

Lets consider a postreceptoral cell type in the retina network capable of estimating achromatic information from the responses of cones in the mosaic⁴. Our model shows ([Alleysson, 2010](#)) that the receptive field of this cell should be weighted to compensate for the variations in the chromatic neighborhood. Without this normalization step, achromatic information would change according to the local proportion of L-, M- and S-cone. This would prevent achromatic information to be represented as a unique projection across spatial dimension. At least for the Parvocellular pathway, this incoherence in spatial reconstruction of achromatic information would increased neural noise ([Alleysson, 2010](#)).

³An image acquired across a mosaic of photosites with different chromatic sensitivities

⁴Horizontal cell are good candidate for such estimation because they have a large dendritic field connecting several cones and they do not show a chromatic-specific response ([Dacey et al., 1996](#))



Figure 3: Illustration of the reconstruction of an image from the mosaic with a wrong demultiplexing function. Colored blobs show up that resemble the phosphenes described by blind peoples equipped with artificial retinas Zrenner (2002)

It is an open question as to whether the normalization is fixed over a lifetime or changes dynamically over a short period of time (Neitz et al., 2002). A dynamic process would allow not only an appropriate projection following the particular neighborhood, but also an adaptation to any modification in the eye’s transfer function. The opposite figure: A fixed weighted solution, poses the question of how the weighting is calculated. Is it learned by experiences (Wachtler et al., 2007) as vision evolves from birth to young adults remaining fixed thereafter? or is it coded genetically by a Markovian decision process as suggested for the construction of the mosaic (Hagstrom et al., 1998, 2000; Knoblauch et al., 2006)?

If we assume that the achromatic information can be estimated by a particular neural sub-network, the chromatic information would be given as a difference between the mosaic output and the achromatic estimate as shown in Figure 2. However, this chromatic information is subsampled and modulated according to the mosaic. The visual system must then be able to demultiplex that information. For this, the spatial position of the cone should be exactly known at some stage of visual processing. What could be the consequence of a false knowledge of the spatial position of each chromatic mechanisms? As we showed in Alleysson (2010) and illustrated in Figure 3 we can simulate an inaccurate demultiplexing of the chromatic channels. The simulation works as follow, we subtract the achromatic map from the mosaic image, demultiplex (by multiplying with the sampling functions) the resulting chrominance map with a new random mosaic, interpolates it and add the achromatic map back.

A bad demultiplexing generates blobs of false color in the scene. This is may be why when blind people are implanted with artificial retina only some phosphenes (Zrenner, 2002) appear in their vision. It is thus highly possible that efficient retinal stimulation requires a knowledge of the patient’s mosaic because, at least until V1, the receptive fields of post-receptoral cells depends on the arrange-

ment of cones in the mosaic. This dependency does not only weight on chromatic vision but also on achromatic vision.

As shown in Figure 3, in the regions of the scene where the activities of the cones are equals ($L = M = S$) the scene is achromatic in the sense of cone sensitivities. In those regions spatial coherence is preserved and the mosaic arrangement doesn’t matter because the response of the three cones are equals (e.g., the clouds in Figure 3). In the hypothesis that the decoding of the mosaic is learned in the first years of age, achromatic region of a scene could be perceived even by newborn without artifacts. This could explain why newborn have difficulties perceiving color stimuli (Adams and Courage, 1998; Mazens et al., 2010).

Neurogeometry operates on an achromatic map but, in daylight conditions, the construction of this map is precluded by the random arrangement of the cone. For this reason midget postreceptoral cells should adapt to the local neighborhood either deterministically or dynamically. We will come back to these issues in Section 4. Another source of constraint in the visual system steem from the regulation of incoming light by the visual system. As we show in the next section, photo-transduction also imply a renormalization process.

3. Non-linearity in color

There are several evidences that perception is a non-linear function of the strenght of the stimulus (Weber, 1846; Fechner, 1889). The visual representation we have of the physical quantities is not a direct but a transformed quality. Concerning the mechanisms that allow the perception of light, we can relate perception (an internal state of representation inside the brain) to physical quantity as already proposed by Fechner (1889). Let suppose that a function f is characteristic of the perception p given a physical quantity ϕ , such that:

$$p = f(\phi) \quad (1)$$

In general, finding f is too much complicated because it is difficult to ask an observer what he/she perceive from the complete space spanned by the physical quantity. Rather, it is common to determine a differential threshold, that is, the amount of change in the physical quantity from ϕ_1 to ϕ_2 for which an observer sees a difference. One can therefore characterize f from its derivative f' through the variation in perception $\Delta p = f'(\phi)\Delta\phi = f'(\phi)(\phi_2 - \phi_1)$ when ϕ_1 and ϕ_2 do not differ too much (i.e. $\Delta\phi \rightarrow 0$).

It is still unknown whether the non-linearity steem from an optimization of the representation of the physical variable (de Vries, 1943; Rose, 1948) or if it is a consequence of the constraints imposed by biological processes (Baylor et al., 1979) or both. Whatsoever, it is likely that these sources of non-linearities determine visual perception (Bachy et al., 2010, 2012). This is another difficulty for neurogeometry because one of its goals is to explain

perception from the geometry of the space spanned by the internal representation of visual information. It is obvious that the non-linear processing of color information changes the geometry of the brain representation of color expressed in physical domain⁵. But it is still not known what type of non-linearity is the best to predict perception of color from the chromatic stimulation of cones. The difficulty is even greater considering that the visual system adapts to environmental conditions such as global/local photometric context or spatial and temporal frequency components of the scene. It is thus possible that the non-linearity changes with parameters of the viewing conditions.⁶

Depending on the point of view adopted, either physical, biological, psychological, from information theory or colorimetric method, different types of nonlinearities were found. In the following sections we will review the different approaches justifying a non-linear processing of color and its modeling.

3.1. Several nonlinearities in intensity coding

For physics, when considering the minimal photon number to generate vision, several authors agreed that only few (2-3 Velden (1944) or 5-7 Hecht et al. (1942)) are needed to enable the perception of a flash light. Even more, a single photon is enough to trigger a rod because in those experiments a single rod catches only a single photon (Hecht et al., 1942; Velden, 1944; Rieke and Baylor, 1998). In this regime, a function f of kind $p \approx k\sqrt{\phi}$, where k is a constant and ϕ represent the intensity of the flash in number of photons, is suggested. This function actually corresponds to the best signal estimator for the Poisson's noise distribution observed in photon's flux (de Vries, 1943; Rose, 1948).

With regards to biology, i.e., the transduction of radiation by photoreceptor (rod and cone), the dynamic of the amplification of photon catch is given by the kinetics of the concentration of cGMP (cyclic guanosine monophosphate) that opens and closes channels corresponding to a cascade initialised by a photoisomerisation. In that case the current I delivered by a photoreceptor is given by $I = I_{max} \frac{G^3}{G^3 + G_{1/2}^3}$ where G is the concentration of cGMP and $G_{1/2}$ is the half saturation concentration (Rieke and Baylor, 1998). A similar non linear function is found when studying the maximum response of a photoreceptor following an adaptation level. Naka and Rushton (Naka and Rushton, 1966) found an exponent 1 instead of 3 when measuring transduction in fish. They also introduced the fact that the value of input giving the half maximum of response can change with the adaptation state of the photoreceptor. In general, given an adaptation state x_0 , and an excitation level x , the transduction level, y , of a photoreceptor is given by $y = \frac{x^n}{x^n + x_0^n}$. Because neuron cannot

code any range of input, the saturation property of this law is an important quality to include when modeling vision. The adaptation level x_0 , which sets the functional range of the photoreceptor model, is a mechanism by which vision can be successful over many orders of magnitude of intensities.

In psychology, the demonstration by Pierre Bouguer that an adaptive gain applies for lightness perception was shown experimentally for the perception of weight by Weber (1846)). When applied to visual perception, Weber's law states that on any uniform background of intensity I , the threshold ΔI for the detection of a steady pedestal changes in radiation intensity is given by $\Delta I = kI$. So, if $\Delta p = f'(\Delta I)$, as Fechner stipulates, then for Δp to reach a threshold of perception following difference ΔI it must be that $\Delta p = k \Rightarrow \Delta p = \frac{\Delta I}{I}$ then $p = c \log I$ (Weber-Fechner law).

When considering the transmission of visual information from the retina to the brain, information theory shows that it is important to minimize the distortion error between input and output. Under a maximum likelihood hypothesis, models such as the pleistochrome proposed by MacLeod and von der Twer (2003) would give a law in $y = x^{(1/3)}$. Finally, this kind of law is also used for the colorimetric definition of the uniform color space (e.g., CIE-Lab) but without a clear justification.

Depending on the point of view considered the non-linearity in neural coding of radiation differs. Psychophysics shows a precise relation between physical variable and perception. But the law describing the biological mechanisms of transduction, which implement a part of the relationship between physics and perception, is only loosely related to the psychophysical law. In Grind et al. (1970), Koenderink et al. (1970), for example, it is shown how the Vries-Rose non-linear mechanism, which takes into account shot noise, becomes Weber's law as the illumination becomes stronger. According to the authors, the retina network layers imply a change in processing when energy is increased. However, considering the biological mechanisms, Shapley and Enroth-Cugell (1984) found a different law when they studied the relationship between retina gain control and the modulation of spatial and temporal behavior of retinal filtering. One way to reconcile those approaches supposes that these nonlinearities show a different regime of retinal functioning adapted to different conditions of illumination, spatial and chromatic behaviors.

3.2. Multiple components non linearity

Up to now we have only discussed achromatic variations of the physical variable, that is uniformly changing with the intensity level of each chromatic components. When considering perceptual spaces with multiple dimensions, such as trichromatic vision, assigning the correct metric becomes non-trivial. How do the different chromatic components combine together in a system where a differential threshold apply?

⁵There are a lot of color illusions that take as origin non-linear local processing such as simultaneous contrast

⁶This is may be why color appearance models developed today are very complicated and deal with several components describing the visual environment (Fairchild, 2005).

This question can be illustrated with a multidimensional model⁷ as follow: Let L , M and S be the response of L, M and S cones to a particular stimulation. If a non linearity \mathbf{f} apply to these responses, we can write:

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) \quad (2)$$

$$\mathbf{x} = \begin{bmatrix} L \\ M \\ S \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} \ell \\ m \\ s \end{bmatrix} \quad \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(L, M, S) \\ f_2(L, M, S) \\ f_3(L, M, S) \end{bmatrix}$$

$$p^2 = \|\overrightarrow{OP}\|^2 = \mathbf{y}^T \mathbf{y} = \ell^2 + m^2 + s^2 = \mathbf{f}^T(\mathbf{x}) \mathbf{f}(\mathbf{x}) \quad (3)$$

In this multivariable model the perception p is given by the square root of the sum of the square of the components. This is an Euclidean's metric (in (ℓ, m, s) space) where absence of perception, $p = 0$, is found when all the cone responses equal zero. When any of the components are different than zero, the perceptual measure is given by the distance between the point P of coordinates (ℓ, m, s) , corresponding to the light and the origin $O = (0, 0, 0)$. Actually, this distance in the Euclidean's metric correspond to the length of the vector joining these two points. An absolute threshold would then be given by the vector length at which detection occurs. If we assume that in the perceptive domain⁸ (ℓ, m, s) , the threshold is fixed independently of the variables the loci of absolute threshold should be represented by a sphere in (ℓ, m, s) space. This constraint of homogeneity also apply to the differential thresholds from any point of the space and the loci of differential thresholds should also be spherical in the whole perceptual space.

If the non linearity \mathbf{f} was known, we could compute the loci of threshold in the physical space by inverting the model. Using psychophysics, we can figure out the shape of the loci of differential threshold in physical space. This shape is actually the representation of the inverse of the non linearity in vision \mathbf{f}^{-1} if it is applied to a sphere.

Restricting to two dimensional space for simplification, in our example, a spherical perceptual loci (the curve described by a point P of coordinates (ℓ, m)) around a reference point $P_c = (\ell_c, m_c)$, could be written as follow:

$$p^2 = \|\overrightarrow{PP_c}\|^2 = (\ell - \ell_c)^2 + (m - m_c)^2$$

$$\Leftrightarrow \begin{cases} \ell(\theta) &= \ell_c + p \cos(\theta) \\ m(\theta) &= m_c + p \sin(\theta) \end{cases}$$

$$\Leftrightarrow \begin{cases} L(\theta) &= f_1^{(-1)}(\ell_c + p \cos(\theta)) \\ M(\theta) &= f_2^{(-1)}(m_c + p \sin(\theta)) \end{cases} \quad (4)$$

Figure 4(a) represents the form of the threshold in physical quantity L and M for several non linearities \mathbf{f}

⁷This model is similar to the so called "line-element" model (Vos and Walraven, 1972; Guth and Lodge, 1973; Guth et al., 1980; Wyszecki and Stiles, 1967; Koenderink et al., 1972), see also section 3.3

⁸According to Petitot (2008), the perceptive domain is not perception *per se* but the space of the macro-emerging level where perception takes place

upon the hypothesis of the model. For the simulation we have used $a = 1/5$, $L_c = 1$, $M_c = 1$, $\ell_c = f_1(L_c)$, $m_c = f_2(M_c)$. For the black curve we used $f_1(x) = f_2(x) = x$, for the red curve $f_1(x) = f_2(x) = x^{1/2}$, for the green curve $f_1(x) = f_2(x) = \log(x)$ and for the blue curve $f_1(x) = f_2(x) = \frac{x}{x+x_0}$ with $x_0 = 1$. As the Figure 4(a) shows⁹, the locus of just noticeable difference (JND) in physical space is a complex shape because the non-linearity is strong and influences locally the threshold.

In general, it is believed that when the variation of the physical quantity is very small the variation of perception is also very small. In that case, the model of predicting loci of just noticeable differences in the physical space could be written:

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) \xrightarrow{diff} d\mathbf{y} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial L} & \frac{\partial f_1}{\partial M} & \frac{\partial f_1}{\partial S} \\ \frac{\partial f_2}{\partial L} & \frac{\partial f_2}{\partial M} & \frac{\partial f_2}{\partial S} \\ \frac{\partial f_3}{\partial L} & \frac{\partial f_3}{\partial M} & \frac{\partial f_3}{\partial S} \end{bmatrix}}_{\mathbf{f}' } d\mathbf{x}$$

$$p^2 = \mathbf{y}^T \mathbf{y} \xrightarrow{diff} 2p = (d\mathbf{y})^T d\mathbf{y} = (d\mathbf{x})^T \mathbf{f}'^T \mathbf{f}' d\mathbf{x} \quad (5)$$

$$d\mathbf{x} = [L - L_c \quad M - M_c \quad S - S_c]^T$$

$$d\mathbf{y} = [\ell - \ell_c \quad m - m_c \quad s - s_c]^T$$

where L_c , M_c and S_c are the cone responses of the light used as a reference of the ellipse (the center of the ellipses)¹⁰. $L - L_c$, $M - M_c$ and $S - S_c$ are the differences in cone responses. \mathbf{f}' is the Jacobian matrix of the vectorial function \mathbf{f} which contain partial derivative¹¹. It is clear from Equation 5 that the resulting loci of JND is an ellipsoid¹². Actually, this schema is a direct extension of the Fechner's principle of determining the function of perception from its derivative in the multidimensional case. But here, the non-linearity of perception is approximated linearly such that the shape of the discrimination loci is an ellipsoid.

Because shape in Figure 4(a) are not ellipsoid, there is a difference in the loci estimated directly by guessing the non-linear function \mathbf{f} and the ellipsoid's shape obtained using the linear approximation of the function using derivative (Equation 5). Actually, the difference is only local because along the color space the ellipsoid's shape will follow the shape obtained with \mathbf{f} .

⁹Remember that we use the inverse of the function to compute the curves

¹⁰It is usually believed that LMS responses are linearly related to CIE-XYZ because CIE data are computed from color matching experiment. Two color are matched when their lights are perceived as equal in spite of differences in their spectral composition. This phenomenon of metamerism is assumed to occur when cone responses are equals, hence linearly related to CIE coordinates (Koenderink, 2010).

¹¹For simplification we omit that partial derivative in Equation 5, $\frac{\partial f_1}{\partial L}$ should be read $\frac{\partial f_1}{\partial L}|_{(L_c, M_c, S_c)}$. Otherwise said it is the linear approximation (through the linear tangent space) of the non linear function f_1 at the point of the center of the ellipsoid.

¹²It is actually true if $\mathbf{f}'^T \mathbf{f}'$ is symmetric definite positive.

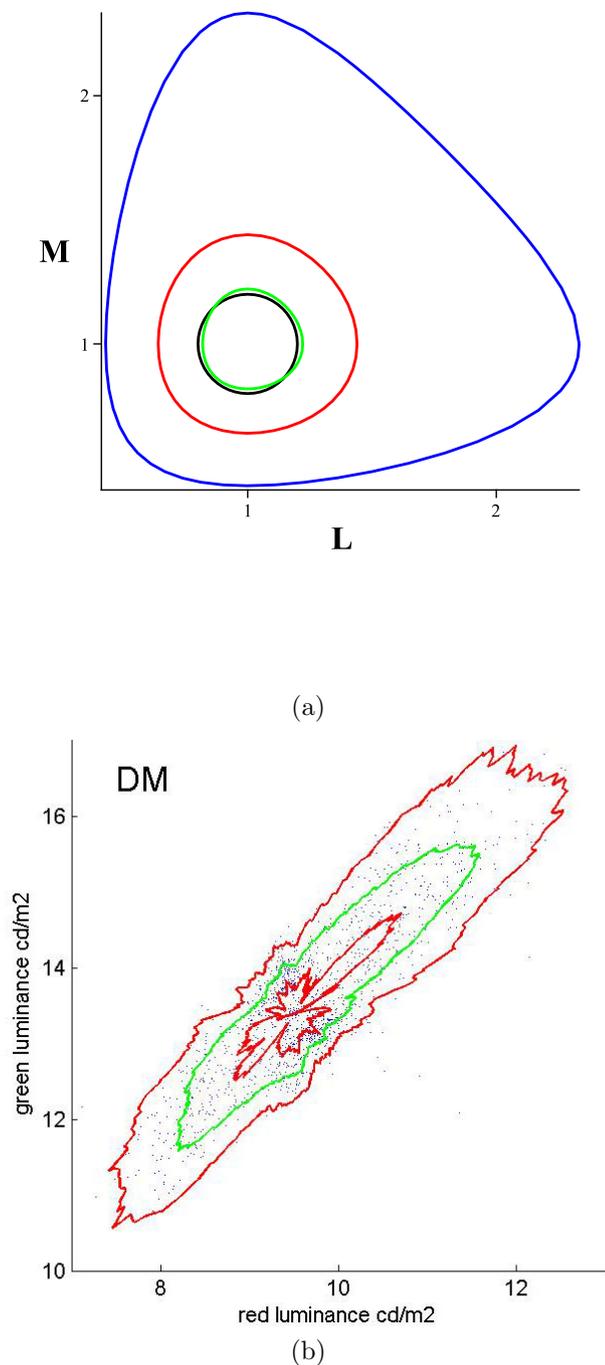


Figure 4: (a) Several loci of just noticeable difference in chromatic intensity for a model of perception in $\sqrt{\cdot}$ (red), \log (green) and $\frac{x}{x+x_0}$ (blue). The loci are calculated with the inverse of those functions. (b) Estimated loci (green line) for one observers in our experiment against a black background. The blue points represent the *identical* responses. The *different* responses are not shown. The red lines gives the standard deviation (± 1 SD)

But, the assumption that ellipsoid's shape is the loci of color discrimination is generally considered true (Silberstein and MacAdam, 1945; Wandell, 1982). It is a strong assumption because it implies that the metric is quadratic (i.e. of order 2) and that we can use the first derivative of the vectorial visual function as a model of the visual system. Actually, Sankeralli and Mullen (1996) found a metric of order 4 whereas Cole et al. (1993) found a metric changing from observer to observer and being in average of 2.3. Non elliptic shapes have also been described by Para (1970); Parra (1972); Benzécri (1981), who measured color discrimination against a black background. Our estimation of the loci of JND (See Annexe A for the detail of the measure) displayed in Figure 4(b) show also a non elliptic shape. These results suggest that the transformation from physical to perceptual spaces is not well approximated with a first derivative of \mathbf{f} under a quadratic metric.

To date, there is no model which predicts these loci, precisely determining the non linearity in the visual system and seeing whether it conciliate models of biological processing with physical one. In the example of Figure 4(b), several hypotheses could be followed. The global shape is elongated approximatively along (R+G) and (R-G) axis (which is probably not the L and M axis, thus a rotation is also needed in LM space), which suggest that the visual function \mathbf{f} is vectorial, having non-vanishing derivatives along each dimension. Vectorial function potentially draws any shape. But the metric and/or the non-linearity should be modified in order to fit the particular shape shown in Figure 4(b). As an example, we share with Parra the fact that the threshold is measured against a black background. Because of that, the adaptation state of the observer is not fixed. A floating adaptation state could lead to local modifications of the geometry. In such an hypothesis, the adaptation state would depends on the test stimuli and the shape would no longer be an ellipse.

3.3. Experiment and model for color discrimination

The best known measure of discrimination thresholds in color is the one performed by MacAdam (1942). MacAdam tested the accuracy of color matching in a particular color space (xyl) while varying intensities in various directions. As shown in Figure 5(a) he found ellipses in the chromatic diagram CIE-xy. It is particularly important to model these ellipses since by definition, a uniform color space is a space where the JND are constant for any direction.

The model introduced by Helmholtz (1896), the so-called line elements model, approaches the non-linear function of perception by its linear approximation around a particular point at which the discrimination is measured. Line elements means that the measure of distance between two stimuli that appear just noticeably different is done along a straight line. Several authors have tried to reproduce elliptic discrimination contours with this model (Wyszecki and Stiles, 1967; Schrödinger, 1920; Stiles, 1946; Vos and Walraven, 1972; Koenderink et al., 1972).

In their paper, Koenderink et al. (1972) summarize the line elements of Helmholtz-Stiles and Schrödinger. The following equation encompass those models:

$$p = (L_c + M_c + S_c)^{\alpha-2} \left\{ \frac{dL^2}{L_c^\alpha} + \frac{dM^2}{M_c^\alpha} + \frac{dS^2}{S_c^\alpha} \right\} \quad (6)$$

Koenderink proposes that $\alpha = 0$ for his model, $\alpha = 2$ and $\alpha = 1$ for Helmholtz and Schrödinger's models, respectively. It is interesting to compare the prediction of these models even if, according to Koenderink et al. (1972), it is senseless to perfectly fit the MacAdam ellipses because they are not representative of many observers in many circumstances. It is more fruitfull to consider whether these metrics (and the one proposed by Guth et al. (1969); Guth and Lodge (1973); Guth et al. (1980)) agree with Abney's additivity law or Bezold-Brücke hue shift (Koenderink et al., 1972). As show in Figure 5(a) the different models (green ellipses for Koenderink, red for Schrödinger, black for Helmholtz) do not fit perfectly MacAdam's data. This shows that the proposed models are not sufficient. Moreover, two of these models are not compatible with the derivation proposed in Equation 5. Equation 6 could be rewritten in a form of Equation 5 if:

$$\mathbf{f}' = \begin{bmatrix} \frac{(L_c+M_c+S_c)^{\frac{\alpha-2}{2}}}{L_c^{\alpha/2}} & 0 & 0 \\ 0 & \frac{(L_c+M_c+S_c)^{\frac{\alpha-2}{2}}}{M_c^{\alpha/2}} & 0 \\ 0 & 0 & \frac{(L_c+M_c+S_c)^{\frac{\alpha-2}{2}}}{S_c^{\alpha/2}} \end{bmatrix} \quad (7)$$

which mean that the partial derivative of f_1 along L direction, $\frac{\partial f_1}{\partial L}$ equal $\frac{(L_c+M_c+S_c)^{\frac{\alpha-2}{2}}}{L_c^{\alpha/2}}$. In the same time, the partial derivative of f_1 along M is equal to 0. But

$$\frac{\partial}{\partial M_c} \int_{L_c} \frac{(L_c + M_c + S_c)^{\frac{\alpha-2}{2}}}{L_c^{\alpha/2}} dL_c \neq 0$$

So, in general, these line elements don't correspond to the Jacobian matrix of a stipulated non-linearity. Only the value $\alpha = 2$ (Helmholtz's proposal) is compatible with a trichromatic model of line element. For the two other models it may be postulated that the threshold is calculated in a 4-dimensional perceptual space, where the fourth dimension $A = (L + M + S)$ is achromatic. This poses the question of how achromatic component is estimated in the visual system as discussed previously (Section 2).

We also proposed a model of MacAdam ellipses based on the Naka-Rushton's law (Alleysson and Héroult, 1998, 2001). Because this law apply to the transduction of cones, saturates and include an adaptation factor¹³ it seems more appropriate than any non-adaptative model. Our model

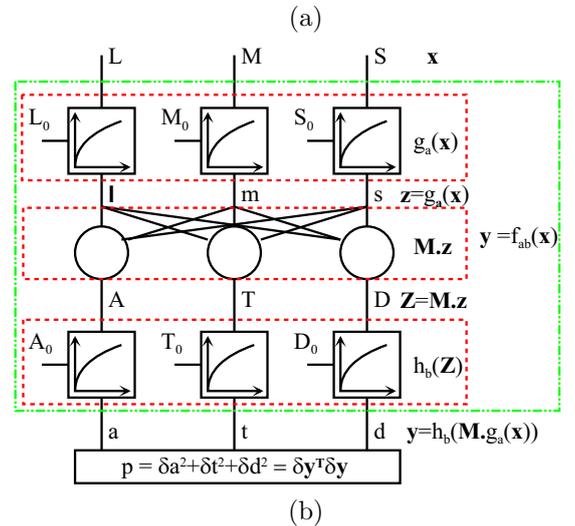
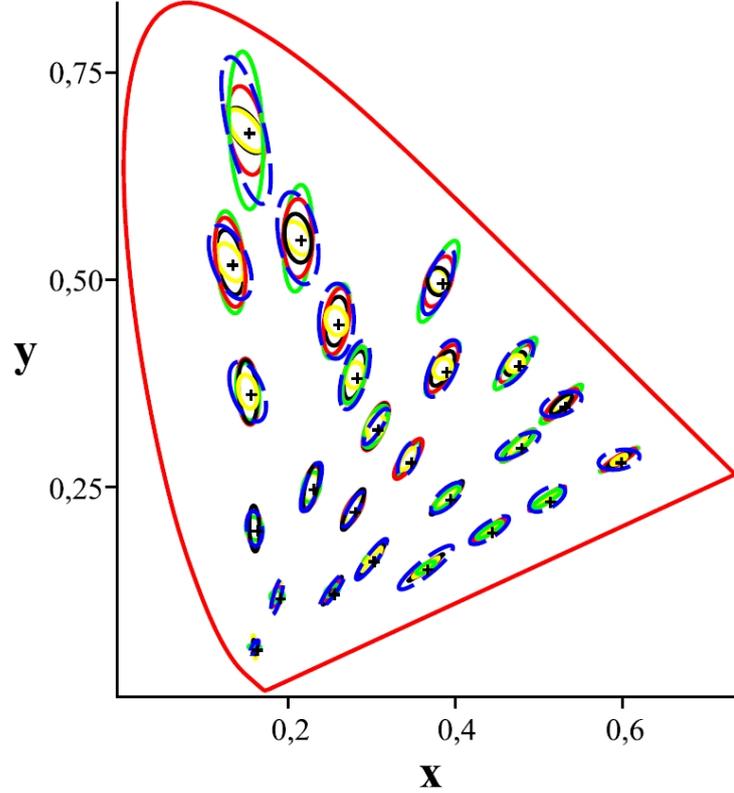


Figure 5: (a) Representation of MacAdam ellipses in CIE-xy space (blue curves). Model of ellipses as proposed by Koenderink (green curves), Schrödinger (red curves), Helmholtz (black curves) and our (yellow curves) (b) The three layer model of color vision

¹³We think of an adaptation factor as an important mechanism to be able to modulate the color non linearity following the adaptation state of the oberver. It allows modulate geometry of color vision following adaptation.

is very similar to the one of Helmholtz (see Figure 5(a) ellipses in yellow compare to ellipses in black) because it also involve a $1/x^2$ function and does not imply a fourth achromatic dimension of input. Even if we have an additional adapting parameter in addition to the metric of Helmholtz: as MacAdam does its experiment on a fixed background we choose this parameter constant, the model resemble closely that of Helmholtz. The Naka-Rushton's law is very similar to the log law found in the Weber-Fechner model and it's adaptation factor account with chromatic adaptation as described by Kries (1902) and Alleysson and Süssstrunk (2002).

Recent findings on retinal modeling for discrimination suggest that a three layer model should be taken into consideration (see Stockman and Brainard (2010) for a review, Webster and Mollon (1991); Zaidi and Shapiro (1993); Chichilnisky and Wandell (1999); Alleysson and Héroult (2001)). The emerging model, as show in Figure 5(b) is composed of an initial non linear layer which regulate the signal representing the quantum catch to allow for a high range of encoding. The second layer is linear, simulating the coding into achromatic and two opponent chromatic components. Finally, the third layer called the desensitization site (Yeh et al., 1993) is also non linear, adapting to contrast (Webster and Mollon, 1991). In these models, the Jacobian matrix in Equation 5 is written as the product of three Jacobian matrices, one for each layer, because layers are assumed to be independent. Another postulate is that non linearities apply separately on each channel and that only the linear part is combined in the different channels. This simplifies the model as the Jacobian matrices for the two non linear layer are diagonal (Alleysson and Héroult, 2001).

Model of the geometry of color space is found to vary a lot with the spatial and temporal property of the stimuli (Noorlander et al., 1981; Noorlander and Koenderink, 1983). Thus, a model of color perception based on a geometry that take only the coordinate of color in a color space, ignoring adaptation, spatial and temporal processing would certainly be incomplete. As an illustration, we recently proposed a model of image processing that take into account adaptation modulated by the spatial content of the scene (Meylan et al., 2007). As shown in Figure 6 the rendering of the raw image coming from a digital camera through spatial driven local adaptation plus demosaicing is far better than the classical rendering through demosaicing plus global tone mapping.

What are the non linear process that shape our vision of color? Mostly, studies of color vision and color discrimination are performed on uniform field in controlled laboratory conditions. Viewing natural scene is far more complex and the plasticity of visual functions is not easily described in terms of modulation in the spatial, temporal and chromatic dimensions.



Figure 6: Rendering of a raw image coming from digital camera (a) with demosaicing followed by global tone mapping (gamma correction) (b) spatially driven local tone mapping followed by demosaicing (Meylan et al., 2007)

4. Neurogeometry of color vision

It is not clear that a neurogeometry of color vision can easily emerge. From the points discussed in section 2 the application of neurogeometry designed for the perception of form in achromatic domain could not be extended easily because of the cone mosaic sampling that generate mixed spatio-chromatic representation of visual information. It seems difficult to design a dynamic processing that would be compatible with the mosaic representation of visual information. However, there are some evidence today that the plasticity of the visual system allows a fast adaptation to the environment (Neitz et al., 2002; Yamauchi et al., 2002), which can be as fast as the dynamic process put forward by neurogeometry. It would be very interesting to see if any fibration, based on $\mathbf{R} \times \mathbf{C}$ but applying on a spatial multiplexing of color \mathbf{R} could achieve demosaicing.

From the point of view of geometrical representation of the color space spanned by the response of neuron in the retina, the non linearity of the chromatic processing does not show any agreement on which the model of perception could be build on. The strong curvature induced by the non linearity could prevent the construction of a model based on differential geometry to match the experimental data. Also, adaptation seem to play a major role in the construction of the geometry. The color geometry may change according to the adaptation state of the observer. In the following paragraph we develop several solutions for a neurogeometry of color vision.

4.1. Reconstruction of Achromatic/Chromatic component before V1

A condition for a direct application of the neurogeometry of form perception on achromatic input, at the level of V1, is that achromatic/chromatic processing is already done at that stage. In a previous paper (Alleysson, 2010) we illustrate how difficult it is to separate achromatic and chromatic components at the spatial frequency where they are aliasing. It probably requires an iterative process using feedback about contour from the coarse estimate of achromatic component carried by the magnocellular channel.

For chromatic processing it is required that each component be grouped together to allow interpolation and comparison from component to component. We think that

this rearrangement occurs in the projection from retina to the primary cortex through Lateral Geniculate Nuclei (LGN). At the output of the retina, ganglion cell which project their axons to the LGN still show a multiplexing of L-M component with achromatic component for midget type (Ingling Jr and Martinez-Uriegas, 1985). Also, midget ganglion cells carrying L-M component are still spatially multiplexed with bistratified ganglion cells carrying S-(L+M) component (Dacey et al., 1996). LGN shows several layers (Martin et al., 1997; Chatterjee and Callaway, 2003) identified as Magnocellular, Parvocellular and Koniocellular, where each layer contains a particular chromatic coding respectively, coarse achromatic, L-M plus fine achromatic and S-(L+M). At the level of V1, color is already coded by blobs separated from achromatic blobs (Livingstone and Hubel, 1987). We speculate that each chromatic component could also be separated up to this level. This observations suggest that the LGN play an important role in the separation of achromatic versus chromatic information and in isolating each chromatic component.

A neurogeometry of form at the level of V1 could provide a template, a raw contour estimation, which could be used by V1 to process the Magnocellular signal and improve the quality of separation of achromatic and chromatic component either in V1 or by feedback to the LGN. Finally, we wouldn't exclude the possibility that the separation of achromatic and chromatic component and the interpolation of chromatic component is integrated in the dynamic process of perception. In both these hypotheses chromatic perception of objects' surface would be influenced by contours. As an example, it is noticeable that illusory contours may also act as a real contour for figure/ground separation which would explain why Kanisza's triangle appear at the forefront. The same interpretation may apply to the filling-in effect show for achromatic (Craik-O'Brien-Cornsweet illusion) or for colored surface (Watercolor or Neon color spreading illusion) (Feitosa-Santana et al., 2011; Pinna and Grossberg, 2005).

4.2. Non linearity in color difference considering the mosaic

In their papers, Noorlander et al. (1981); Noorlander and Koenderink (1983) show color discrimination measurements for stimuli that have several temporal and spatial frequency tuning. They found that the ellipses for high spatial frequency are very different than those for lower spatial frequency. This is may be the sign that the mosaic acquires an importance when the resolution of the stimuli becomes higher.

In any case, it is clear that at the scale of the photoreceptor, the mosaic should have an influence on visual information coding. However, this does not imply that the mosaic sampling could be the reason for non linearity in color vision. It is suggested by previous work that achromatic non linearity is a consequence of Poisson distributed photons at low levels of luminosity and visual gain control at higher levels (see Section 3). Whether this suggestion

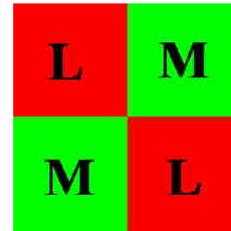


Figure 7: An example of mosaic pattern

holds when considering the mosaic sampling is an open question. Indeed, one of the implicit assumption in the line element theory of color discrimination is that all cone responses are available at all positions. This could be reasonably true when considering a large uniform stimuli as input. But when considering a stimuli as tiny as the cone pattern in the mosaic, the local arrangement in the mosaic should be taken into account¹⁴.

As an illustration, let suppose that we consider a 2x2 mosaic with a particular arrangement of L- and M- cone (Figure 7). We suppose that achromatic information is estimated from the response of these cones by an horizontal cell (Alleysson, 2010) whose receptive field is restricted to this particular patch of the mosaic.

If we suppose that the response of cones are processed through a non linear function (the same function is supposed to apply for L and M mechanisms) and that the achromatic function is estimated as a linear combination of L and M cones, we can write:

$$\ell = f(L) \quad m = f(M) \quad A = L + M \quad (8)$$

$a = g(A)$ implies $a = g(L + M) \neq f(L) + f(M)$ in general. The non linearity applied to each cone mechanism is different to applying a non-linearity to a linear combination of both of them. If we take into account the sampling of the retinal mosaic, the linear combinations to which the non-linearity is applied varies from position to position. Thus, the details of the retinal processing shouldn't be ignored when we look for mechanisms of light regulation in the visual system. Furthermore, we are convinced that the mosaic sampling influences the way light is regulated through the mechanisms applied for reconstructing spatially coherent information, even for large and uniform field stimuli.

In the general case, it is necessary to consider that signal modulation by a non linear process should be driven locally by a spatial domain as large as the receptive field of the cell which implements the non-linearity. For the formulation of Equation 5, this means that the function \mathbf{f} would not only depend on L, M and S, (which are not present together at the same position) but also by the neighborhood of cones included in its receptive field. We

¹⁴This is especially true as it has not been clearly demonstrated that the blurring caused by the eyes optics allow us to discard the problems inherent to mosaic sampling.

may consider the non linearity of color vision \mathbf{f} to depend on the responses of cones (even rods) over a large area of the retina. In this case, the model would be written as $p = \mathbf{f}(\{(x_i, y_i) \in \mathbb{D}\})$, where $\mathbb{D} \subset \mathbf{R}$ is a spatial domain of the retina. In that case, the differential space of color vision would depend on many variables, all the partial derivative of \mathbf{f} which depend on all the response of photoreceptor in the receptive field of the cell. In the line element model of discrimination, Schödinger and Koenderink implicitly use an achromatic component (see Section 3) which must be estimated from a neighborhood of LMS response of cone in the mosaic. We believe that the contribution of each photoreceptor in the receptive field, due to the mosaic arrangement, could be included into the contribution of the adaptation coefficient. In this scheme, for a given adaptation state, the geometry would be constrained by both spatial and temporal content of the scene but also by the local arrangement of cones in the neighborhood.

4.3. Quantum Vision

Somehow the visual system transforms the rainfall of quanta caught by the retina, into a continuous and stable representation. In that sense, the visual system is a *quantum processor* (Penrose, 1989), which is able to give us a classical physical representation of the world from quantum physics (Schrödinger, 1935). Contemporary models of vision and color vision use the wave representation of light (Maloney, 1999; Schultz et al., 2006) with its properties of emission, reflexion on object, transparency. But, because the inner nature of light and photon catch by photoreceptor is quantum and because the mosaic sampling is at odd with wave representation of light, we strongly believe that mosaic sampling as something related to the quantum nature of light. But for now, apart for Bouman (2002, 2006) contributions, who shows a kind of quantum spatio-temporal correlation in the visual system, the quantum nature of light is solely revealed in absolute photon threshold experiment by the $\sqrt{}$ function observed in the De Vries-Rose's model. For higher intensities, the differential threshold is given by the Weber-Fechner's law. In their papers, Koenderink et al. (1970) suggest that the function modification is due to the network of postreceptoral cells. They go further in modeling this neural mechanisms as a quantum coincidence counter, extending the question of the quantum nature of light for vision to high luminance situation. But, despite its elegant demonstration, we can only assume, as Bouman (2002, 2006) pointed out, that the nature of image processing in the visual system must be in accordance with quantum physics.

5. Conclusion

Neurogeometry does not apply directly to color vision because the sampling of colors through the random mosaic of cones implies a non trivial projection of the three dimensional color space to a scalar. This projection is not easily

invertible and the representation by neurons of spatial and chromatic component of natural scenes are not fully understood. The random structure of the mosaic modifies the statistics of the neural information sampled. Even if the physics of the visual scene is stationary in spatial, temporal and chromatic dimension, responses of photoreceptor due to the mosaic will not be. The second problem for neurogeometry of color vision is that the dynamics of color from discrimination to phenomenology is not yet modeled and understood completely. We are far from having a clear idea of what is occurring between the physiology of neuron in the visual system and the phenomenology of color perception. Here we propose that considering the interlaced mosaic sampling by cones could be the key for establishing a neurogeometry of color vision. We also suggest that light could be then efficiently handle by a locally driven adaptive non linearity which would apply for both light regulation and demosaicing.

As the popular proverb says, "It is not for the blind to give an opinion on colors". Color vision still resists modeling, mainly because the best way we have to describe geometry of color space, does not correspond to phenomenology. Ideally, best discrimination should occur at the frontiers between color categories. It is still a puzzling result that color categorization does not correlate with discrimination performance (Danilova and Mollon, 2010). In its last chapters, Petitot (2008) claims that neurogeometry could be the missing piece linking local behavior of micro neural level to the macro emerging level. In turn because phenomenology is an isomorphism of this emerging level, this piece would also fill the gap between the sampling of physical space and visual appearance. This could be a solution for color perception too. Pursuing this project, we would favor a description in terms of spatio-temporal chromatic representation rather than the 2D -21/2D -3D sketch proposed by Marr (1982). The chromatic domain should be more accurate and powerful because it must be treated at the scale of neurons. If we want to model perception from neural activity of cells it would be straightforward to consider color.

Finally, beyond these models, the underlying mathematical analysis and physical representation of the world, the issue of neurogeometry would be allowing blind people to see. Explaining visual experience from neural activities would enable us to build artificial vision system that would favorably stimulate the visual representation of an individual.

Appendix A. Estimation of discrimination contours in the Red/Green plane

For any reference color in a color plane (e.g., the R/G plane), comparison colors can be defined in polar coordinates (α, r) centered on the reference. For a given direction of test (α_t) , the radius (r_t) that gives a comparison color perceived by an observer as being different from the reference in a proportion p of the trials (e.g., $p = 50\%$) define a

point of just noticeable difference (JND). A discrimination contour is the 2-D shape described by the JND when it is computed for all possible directions around the reference point. Even when using adaptation method, reliable psychophysical determination of the just noticeable difference requires many trials. For convenience, studies on discrimination contour usually measured the JND for a limited set of directions, typically 12 to 20. Some studies further assumed a symmetry of the contour and selected the direction of JND estimation over a portion of the angular range. In the following sections we present an alternative method for the estimation discrimination contour. We illustrate the approach with preliminary results from the two authors (D.A and D.M).

Appendix A.1. Materials and method

Appendix A.1.1. Apparatus and calibration

Stimuli were displayed on a Dell Trinitron monitor (ultrascan 1600HS) driven by a VSG 2/5F graphics board (Cambridge Research Systems, Rochester, England). The VSG was controlled using the CRS Toolbox for MATLAB (version 1.271) and the VSG 8.107 library. The display was confined to an area of 800×600 pixels with a mean frame rate of 60Hz. Pixel size was 0.405 mm, 0.0232 degree of visual angle at the viewing distance of 1 m. In addition of the built-in gamma correction files provided by the VSG system. The spectra and luminance of the monitor for different beam intensities was carefully checked using an eye-one spectrometer (GretagMacbeth, Switzerland). Monitor luminance output was further linearized directly in software using these measurements.

Appendix A.1.2. Stimuli, task, and procedure

We used our setup to simultaneously display two colored patch over a black background. The stimulus was a circular area (2° of visual angle) cut in two halves separated by a vertical bar filled with the background color (gap width = 0.8°). The reference and comparison color were presented together for 3 seconds with a linear ramp of 0.2 s at the start and end of the display time. The observer viewed the patches binocularly without fixation constraint from a distance of 1 meter. The task consisted in judging if the two patches were different. The forced choices (different or identical) were recorded using a button box with a maximum response time fixed at 6 seconds. The reference color for the two observers was an arbitrary yellow (CIEXYZ-1931, $x = 0.471$, $y = 0.457$) with an average luminance of $Y=22.9$ cd/m². We selected the comparison color using a two step procedure. For the early trials we randomly draw the direction of test (α_t from 0 to 2π) and the distance (r_t from 0 to 3) with the precision of 15 bits as provided by the video card. After a reasonable number of trials (typically 50), we estimated confidence ellipses for the distribution of the comparison points that were not discriminated. In this second step we used large and small confidence values (e.g., 0.99 and 0.25)

to define a confidence area for the JND values. We picked up the next comparison color within this confidence area and repeated the second step after the observer's answer. The rationale for choosing pseudo randomly the comparison angle and radius was to avoid any assumption about the psychometric function during the measurement. The left/right position of the reference and comparison color were chosen randomly on each trial. For 10 percent of the trials the comparison color was equal to the reference and a warning beep occurred if the observer answered 'different'. Measurements were made in several sessions, spaced over two weeks.

Appendix A.1.3. Method simulation

Using confidence ellipses to estimate a confidence area for the JND, we do not assume that the observer's discrimination contour is an ellipse. As shown by the simulation described in this section, accurate estimation of the full 2-D discrimination contour can be done with about 4000 trials even when the modeled observer's contour noticeably deviate from an ellipse. Figure A.8 show the results for 2 discrimination contours with a different shape (black lines). We used the above described procedure to select comparison stimuli. On each trial, the observer's response was simulated using a normal cumulative distribution function with a mean equal to the length of the radius in the direction of the test and a proportional standard deviation (25% of the radius length). The estimated contour (in green) was computed by fitting a psychometric function to the responses. In practice, we ordered the responses according to the direction of test. Starting from the angular value of zero, we binned the first 170 points and estimated the threshold and SD within the spanned angular range. We then moved counterclockwise by 25 points and estimated the new threshold and SD. This was repeated until the end of the response vector.

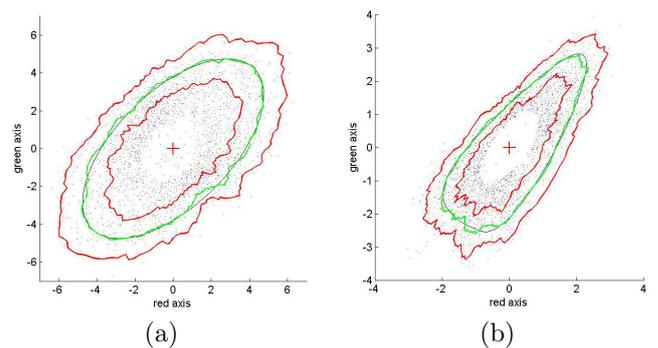


Figure A.8: Modeled (black lines) and estimated (green lines) discrimination contour. The method is successful in approaching the simulated contour for ellipse (a) and for shapes that notably deviate from an ellipse (b). The blue points represent the 'identical' responses of the simulated observer. The 'different' responses are not shown. The red lines give the standard deviation (± 1 SD)

Appendix A.2. Preliminary results

Figure A.9 show our preliminary results. The contours for DM and DA are based on 3600 and 1800 trials respectively. Each JND estimate is based on 140 samples for observer DM and 80 for observer DA. Given the small number of trials, the contour for DA is not yet fixed but the method already show an estimation coherent with the data from DM. The contours are elongated along the luminance axis (R+G) indicating that changes in the chromatic composition of the comparison color are detected more easily than changes in luminance. There is a clear asymmetry between the threshold in the direction of luminance increase compared to the direction of luminance decrease. This asymmetry may stem from the local non linearity of the perceptual color space. The shape contain also some bubbles showing the complicated metric involved in this multidimensional difference estimation. These two elements confirm that a proper metric should be designed for color difference perception and that color non linearity could influence locally the perception of differences.

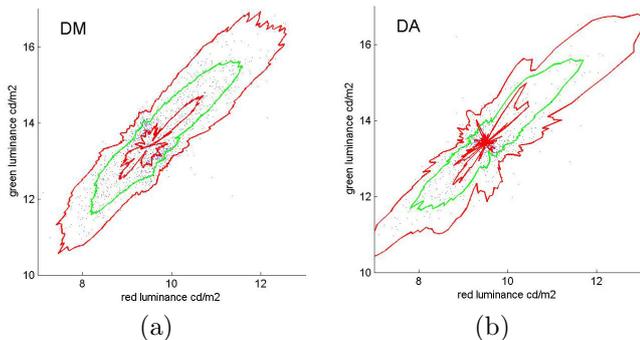


Figure A.9: Estimated (green lines) discrimination contour for the two authors. The blue points represent the 'identical' responses. The 'different' responses are not shown. The red lines gives the standard deviation (± 1 SD)

Appendix A.3. Discussion

The method may be improved but our simulation show that an estimation of the full discrimination contour could be done in much less trials (about 4000) than with a direction fixed method (a 1° resolution for the contour would require about 360×80 trials). The preliminary results show that the shapes of the discrimination contour are elongated but appear more complex than the classical elliptical model. As discussed in chapter 3, a model of the non linearity associated with biological processing in the visual system should be able to fit these contours.

Acknowledgments

D.M. is funded by the French CNRS (ST 997-03-UMR5105). D.A. thanks Seitz Phototechnik A.G. for their financial support. We thank anonymous reviewers for their valuable comment that has helped improving the manuscript. We are also grateful to Lucas Wilkins for his careful checking.

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